Review on Topology

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통계이론세미나 - 위상구조의 통계적 추정, 2023 가을학기

[4] 우리의 철학I. 같은 것은 같도다. (Isomorphism 의 철학)
우리의 철학II. 같은 것은 정말 똑같다. (Identification 의 철학)

Definition. [3, Section 3.1] 함수 $f : X \to Y$와 $x_0 \in X$가 주어져 있을 때, 임의의 $\epsilon > 0$ 에 대하여 다음의 성질

$$x \in X, \|x - x_0\| < \delta \Longrightarrow \|f(x) - y_0\| < \epsilon$$

이 성립하는 $\delta > 0$가 존재하면, 함수 f가 점 x_0에서 *연속*이라 한다.

만일 집합 $A \subset X$의 모든 점에서 f가 연속이면 A 위에서 연속이라 하고, 정의역 위에서 연속인 함수를 *연속함수*라고 한다.

Topological Spaces, Continuous Functions, and Homeomorphisms

Definition ([2, Section 12])**.** A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. If $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.
3. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

Example. Let X be a three-element set, $X = \{a, b, c\}$. All the possible topologies are schematically represented in Figure 1. For example, the diagram in the upper right corner indicates the topology $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. All the topologies can be obtained by permuting a, b, c.

Definition ([2, Section 13])**.** If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that $\emptyset$ and X are in $\mathcal{T}$.

1. For each $x \in X$, there is at least one $B \in \mathcal{B}$ containing x.
   (a) If $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ containing x such that $B_3 \in B_1 \cap B_2$.

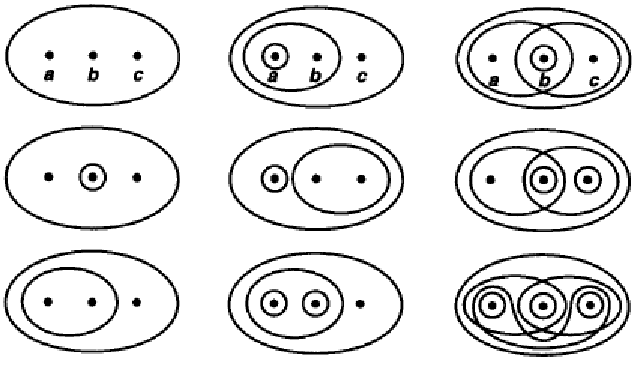

Figure 1: [2, Figure 12.1] Example of topologies of a three-element set.

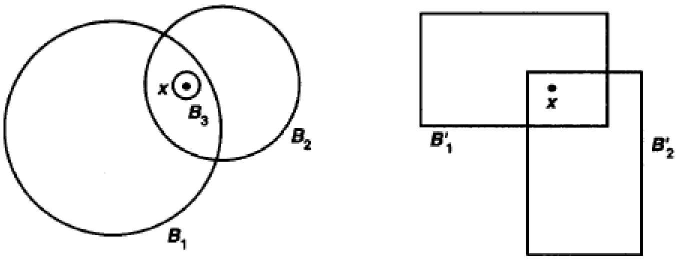

Figure 2: [2, Figure 13.1, 13.2] Example of bases of circular regions or rectangular regions.

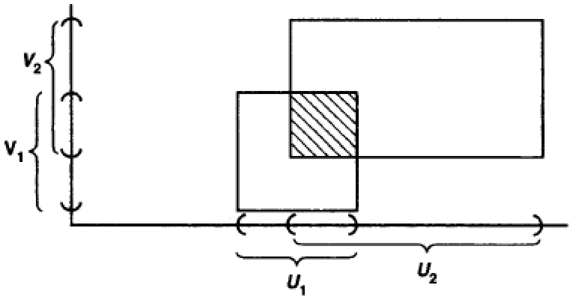

Figure 3: [2, Figure 15.1] Product topology.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane $\mathbb{R}^2$, as in Figure 2 left, then $\mathcal{B}$ is a basis. Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane $\mathbb{R}^2$, as in Figure 2 right, then $\mathcal{B}'$ is also a basis. And in fact, two bases $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology for $\mathbb{R}^2$.

Definition ([2, Section 15])**.** Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having the basis as (see Figure 3):

$$\mathcal{B} = \{U \times V \subset X \times Y : U \text{ is open in } X, V \text{ is open in } Y\}.$$

Remark. This definition of the product topology can be naturally extended to a finite product space $X_1 \times \cdots \times X_n$.

Definition ([2, Section 16])**.** Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.

Definition ([2, Section 17])**.** A subset A of a topological space X is said to be closed if the set $X \backslash A$ is open.

Definition ([2, Section 17])**.** The closure of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Definition ([2, Section 17])**.** We say U is a neighborhood (neighbor) of x if U is an open set containing x.

Definition ([2, Section 17])**.** If A is a subset of a topological space X, We say x is a limit point of A if every neighborhood of x intersects A in some point other than x itself.

Theorem. *Let A be a subset of a topological space X, and A' be the set of all limit points of A, then*

$$\bar{A} = A \cup A'.$$

Definition ([2, Section 17])**.** A topological space X is called a Hausdorff space if for each pair $x_1 \neq x_2 \in X$, there exists neighborhoods U_1, U_2 of x_1, x_2, respectively, that $U_1 \cap U_2 = \emptyset$.

Definition ([2, Section 18])**.** A function $f : X \to Y$ is continuous if for each open set V of Y, $f^{-1}(V)$ is an open subset of X.

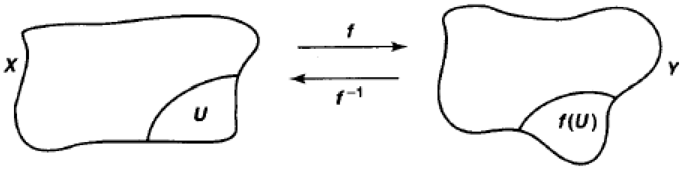

Figure 4: [2, Figure 18.1] Homeomorphism.

Remark. It suffices to show that the inverse image of every basis element is open.

Theorem ([2, Theorem 18.1])**.** *Let X, Y be topological spaces; let $f : X \to Y$. Then the followings are equivalent:*

1. *f is continuous.*
2. *For every closed set B of Y, $f^{-1}(B)$ is closed in X.*
3. *For each $x \in X$ and each neighborhood V of $f(x)$, there is an neighborhood U of x such that $f(U) \subset V$.*

Definition ([2, Section 18])**.** Let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism.

X and Y are homeomorphic if such a homeomorphism f exists, and denoted as $X \cong Y$.

Remark. Another way to define a homeomorphism is to say that $f : X \to Y$ is a bijection such that $f(U)$ is open if and only if U is open (see Figure 4).

Remark. A homeomorphism gives us a bijective correspondence not only between X and Y but also between the collections of open sets of X and Y. As a result, any property of X that is entirely expreesed in terms of the topology of X yields, via f, the property of Y. Such a property of X is called a topological property of X.

Definition ([2, Section 18])**.** Suppose $f : X \to Y$ is an injective continuous, and let $Z := f(X) \subset Y$ be the image of f equipped with the subspace topology. If the function $f' : X \to Z$ obtained by restricting the range of f is a homeomorphism of X with Z, we say that $f : X \to Y$ is a topological embedding (imbedding) of X in Y.

Definition ([2, Section 20])**.** A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the distance between x and y. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-ball centered at x. Sometimes we omit d and write $B(x, \epsilon)$.

Definition ([2, Section 20])**.** If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

A metric space X is a topological space X together with a specific metric d that gives the topology of X.

Example. Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and for $1 \leq p \leq \infty$, we define the p-norm of x by

$$\|x\|_p := (x_1^p + \cdots + x_n^p)^{1/p}$$

for $p \in [1, \infty)$, and $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$. And then the induced distance d_p on $\mathbb{R}^n$ is defined as

$$d_p(x, y) = \|x - y\|_p .$$

All the metrics d_p induce the same topology on $\mathbb{R}^n$ for $1 \leq p \leq \infty$, and this is the usual topology on $\mathbb{R}^n$. This also coincides with the product topology on $\mathbb{R}^n$ as well.

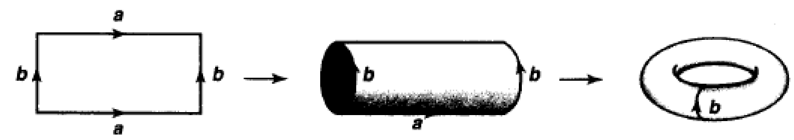

Figure 5: [2, Figure 22.1] Torus as a quotient space.

Theorem ([2, Theorem 21.1])**.** *Let* (X, d_X), (Y, d_Y) *be metric spaces and let* $f : X \to Y$. *Then continuity of* f *is equivalent to the requirement that given* $x \in X$ *and* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$d_X(x, y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

Definition ([2, Section 22])**.** Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is a quotient map if $U \subset Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition ([2, Section 22])**.** If X is a topological space, A is a set, and $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; i.e., $U \subset A$ is open if and only if $p^{-1}(U) \subset X$ is open. $\mathcal{T}$is called the quotient topology induced by p.

Connectedness and Compactness

Definition ([2, Section 23])**.** Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Theorem ([2, Theorem 23.4])**.** *Let* A *be a connected subspace of* X. *If* $A \subset B \subset \bar{A}$, *then* B *is also connected.*

Theorem ([2, Theorem 23.5])**.** *The image of a connected space under a continuous map is conenected.*

Corollary ([2, Corollary 24.2])**.** *The real line* $\mathbb{R}$ *is connected and so are intervals and rays in* $\mathbb{R}$.

Definition ([2, Section 24])**.** Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be path connected if every pair of points of X can be joined by a path in X.

Example ([2, Section 24] Topologist's sine curve)**.** Let S denote the following subset of the plane

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

The set $\bar{S} = S \cup \{0\} \times [-1, 1]$ is a classical example in the topology called the topologist's sine curve (see Figure 6). The set $\bar{S}$ is connected but not path connected.

Since S is a continuous image of $(0, 1]$, S is connected, and then $\bar{S}$ is connected as well. Now we show $\bar{S}$ is not path connected. Suppose there is a path $\gamma : [a, c] \to \bar{S}$ with $\gamma(a) = (0, 0)$ and $\gamma(c) = (1, \sin 1)$. Since $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed in $[a, c]$, it has the largest element b. Then $\gamma : [b, c]$ is a path that maps b into the vertical interval $\{0\} \times [-1, 1]$ and maps $(b, c]$ into S. Since γ is continuous, there exists $\delta > 0$ such that

$$\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5).$$

However, if we write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $\gamma_1[b, b + \delta]$ is a connected subset of $[0, 1]$ containing $\gamma_1(b) = 0$ and $\gamma_1(b + \delta) > 0$, so

$$[0, \gamma_1(b + \delta)] \subset \gamma_1[b, b + \delta].$$

But since $\gamma_2(t) = \sin(1/\gamma_1(t))$ if $\gamma_1(t) > 0$, so

$$\gamma_2[b, b + \delta] \supset \sin(1/(0, \gamma_1(b + \delta)]) = [-1, 1].$$

This contradicts with $\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5)$, so such path γ cannot exist and $\bar{S}$ is not path connected.

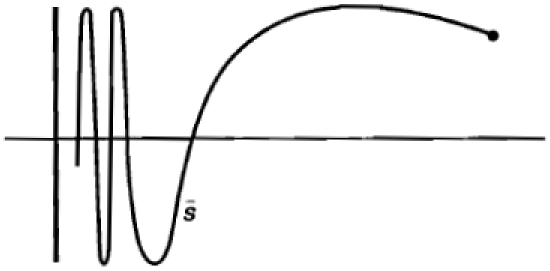

Figure 6: [2, Figure 24.5] Homeomorphism.

Definition ([2, Section 25])**.** Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or the "connected components") of X.

Theorem ([2, Theorem 25.1])**.** *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty conencted subspaces of X intersects only one of them.*

Definition ([2, Section 25])**.** Given X, define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.

Theorem ([2, Theorem 25.2])**.** *The path components of X are path connected disjoint subspaces of X whose union is X, such that each nonempty path conencted subspaces of X intersects only one of them.*

Definition ([2, Section 25])**.** A space X is said to be locally connected at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. X is locally connected if it is locally connected at each of its points. Similarly, a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path connected neighborhood V of x contained in U. X is locally path connected if it is locally path connected at each of its points.

Theorem ([2, Theorem 25.5])**.** *If X is a topological space, then each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.*

Definition ([2, Section 26])**.** A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition ([2, Section 26])**.** A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Theorem ([2, Theorem 26.2])**.** *Every closed subspace of a compact space is compact.*

Theorem ([2, Theorem 26.3])**.** *Every compact subspace of a Hausdorff space is closed.*

Theorem ([2, Theorem 26.5])**.** *The image of a compact space under a continuous map is compact.*

Theorem ([2, Theorem 27])**.** *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the metric induced by p-norm $\|\cdot\|_p$.*

Theorem ([2, Theorem 27] Extreme value theorem)**.** *Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Definition ([2, Section 41])**.** Let $\{U_\alpha\}$ be an indexed open covering of X. An indexed family of continuous functions

$$\rho_\alpha : X \to [0, 1]$$

is said to be a partition of unity on X, dominated by (or subordinate to) $\{U_\alpha\}$, if:

1. (support ρ_α) $\subset U_\alpha$ for each α, i.e., $\overline{\{x : \rho_\alpha(x) \neq 0\}} \subset U_\alpha$.
2. The indexed family {support ρ_α} is locally finite, that is, $\forall x \in X$, there is only finite ρ_α's such that $\rho_\alpha(x) > 0$.
3. $\sum \rho_\alpha(x) = 1$ for each $x \in X$.

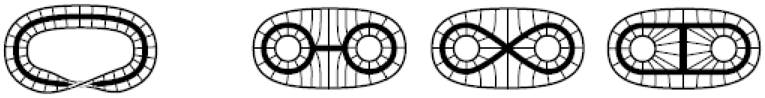

Figure 7: [1, Section 0] Example of a deformation retract.

Homotopy

Definition ([1, Chapter 0])**.** Let $f_0, f_1 : X \to Y$. A *homotopy* between f_0 and f_1 is a continuous function $F : X \times [0,1] \to Y$ such that for all $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Two functions f_0, f_1 are *homotopic* if such F exists, and we write $f_0 \simeq f_1$.

Definition ([1, Chapter 0])**.** A map $f : X \to Y$ is called a homotopy equivalence if there is a map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The space X and Y are said to be homotopy equivalent or to have the same homotopy type, and write $X \simeq Y$, if such homotopy equivalence $f : X \to Y$ exists.

Definition ([1, Chapter 0])**.** Let $A \subset X$. Then A is a *deformation retract* of X if there exists a continuous function $F : X \times [0,1] \to Y$ such that for all $x \in X$ and $a \in A$, $F(x,0) = x$, $F(x,1) \in A$, and $F(a,1) = a$. In other words, A is a deformation retract of X if there exists $r : X \to A$ with $r|_A = id_A$ and r and $id_X : X \to X$ are homotopic. We additionally say A is a *strong deformation retract* of X if F also satisfies $F(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.[1]

Example. See Figure 7. The left figure shows a (strong) deformation retract of a Möbius band onto its core circle. The three figures on the right show deformations in which a disk with two smaller open subdisks removed shrinks to three different subspaces. Also note that these three different subspaces are homotopy equivalent to each other but not homeomorphic.

Definition ([1, Chapter 0])**.** A space X is called contractible if it is homotopy equivalent to a point. This is equivalent to saying that the identity map $id_X : X \to X$ is nullhomotopic, that is, homotopic to a constant map.

Remark. The relationship between different equivalences of topology is as follows:

$$\begin{tikzcd}
\text{X and Y are isometric} \arrow[d, Rightarrow] & \\
\text{X and Y are homeomorphic ($X \cong Y$)} \arrow[d, Rightarrow] & \text{Y is a deformation retract of X} \arrow[dl, Rightarrow] \\
\text{X and Y are homotopic ($X \simeq Y$)} \arrow[d, Rightarrow] & \\
\text{X and Y have the same homology ($H_*(X) = H_*(Y)$)} &
\end{tikzcd}$$

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[1] In [1], strong deformation retract is taken as the definition of deformation retract.