Uniform Integrability and Convergence in L^1

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For this lecture note, we use convergence in probability version of convergences of integrals.

Theorem ([1, Exercise 2.3.4]). *Fatou's lemma. Suppose $X_n \geq 0$ and $X_n \to X$ in probability, then* $\liminf_{n\to\infty} \mathbb{E}X_n \geq \mathbb{E}X$.

Theorem ([1, Exercise 2.3.5]). *Dominated convergence. Suppose $X_n \to X$ in probability and (a) $|X_n| \leq Y$ with $\mathbb{E}Y < \infty$ or (b) there is a continuous function g with $g(x) > 0$ for large x with $|x|/g(x) \to 0$ as $|x| \to \infty$ so that $\mathbb{E}g(X_n) \leq C < \infty$ for all n. Then $\mathbb{E}X_n \to \mathbb{E}X$.*

We will give necessary and sufficient conditions for a martingale to converge in L^1. The key to this is the uniform integrability. To motivate the definition of the uniform integrability, we will see an absolute continuity property of a integrable random variable.

Lemma ([2]). *Let X be an integrable random variable, i.e., a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}|X| < \infty$. Then, given $\epsilon > 0$, there exists a $\delta > 0$ such that for $F \in \mathcal{F}$, $P(F) < \delta$ implies that $\mathbb{E}[|X|; F] < \epsilon$.*

Proof. If the conclusion is false, then, for some $\epsilon_0 > 0$, we can find a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that

$$P(F_n) < 2^{-n} \qquad \text{and} \qquad \mathbb{E}[|X|; F_n] \geq \epsilon_0.$$

But since $|X| 1_{F_n} \to 0$ in probability as $n \to \infty$, [1, Exercise 2.3.5] implies that

$$\lim_{n\to\infty} \mathbb{E}[|X|; F_n] = \lim_{n\to\infty} \mathbb{E}[|X 1_{F_n}|] = 0.$$

and we have arrived at the required contradiction. □

Corollary ([2]). *Suppose that X is an integrable random variable and $\epsilon > 0$. Then there exists $M \in [0, \infty)$ such that*

$$\mathbb{E}[|X|; |X| > M] < \epsilon.$$

Proof. Let δ be as in previous Lemma. Since

$$M P(|X| > M) \leq \mathbb{E}|X|,$$

we can choose M such that $P(|X| > M) < \delta$. □

This leads to the following definition of the uniformly integrable.

Definition. A collection of integrable random variables $\{X_i : i \in I\}$ is said to be uniformly integrable (U.I.) if

$$\lim_{M\to\infty} \left(\sup_{i\in I} \mathbb{E}(|X_i|; |X_i| > M) \right) = 0.$$

Remark. If $\{X_i : \ i \in I\}$ is U.I., then $\sup_{i\in I} \mathbb{E}|X_i| < \infty$. But the converse is not true.

Example. (a) Any finite collection of integrable random variables is uniformly integrable.

(b) A collection of random variables that are dominated by an integrable random variable is uniformly integrable, i.e., if there exists Y such that $|X_i| \le Y$ for all $i \in I$ and $\mathbb{E}Y < \infty$.

Our first result gives an interesting example that shows that uniformly integrable families can be very large.

Theorem ([1, Theorem 4.6.1])**.** *Given a probability space $(\Omega, \mathcal{F}_0, P)$ and an $X \in L^1$, then*

$$\{\mathbb{E}\left[X|\mathcal{F}\right] : \mathcal{F} \textit{ is a } \sigma\textit{-field } \subset \mathcal{F}_0\}$$

is uniformly integrable.

Proof. If A_n is a sequence of sets with $P(A_n) \to 0$ then the dominated convergence theorem ([1, Exercise 2.3.5]) implies

$$\mathbb{E}\left[|X|; A_n\right] = \mathbb{E}\left[|X 1_{A_n}|\right] \to 0.$$

From the last result, it follows that if $\varepsilon > 0$, we can pick $\delta > 0$ so that if $P(A) \le \delta$ then $\mathbb{E}\left[|X|; A\right] \le \varepsilon$. (If not, there are sets A_n with $P(A_n) \le 1/n$ and $\mathbb{E}\left[|X|; A_n\right] > \varepsilon$, a contradiction.)

Pick M large enough so that $\mathbb{E}|X|/M \le \delta$. Jensen's inequality and the definition of conditional expectation imply

$$\begin{aligned}\mathbb{E}\left[|\mathbb{E}[X|\mathcal{F}]|; |\mathbb{E}[X|\mathcal{F}]| > M\right] &\le \mathbb{E}\left[\mathbb{E}\left[|X||\mathcal{F}\right]; \mathbb{E}\left[|X||\mathcal{F}\right] > M\right] \\ &= \mathbb{E}\left[|X|; \mathbb{E}\left[|X||\mathcal{F}\right] > M\right],\end{aligned}$$

since $\{\mathbb{E}\left[|X||\mathcal{F}\right] > M\} \in \mathcal{F}$. Using Chebyshev's inequality and recalling the definition of M, we have

$$P\{\mathbb{E}\left[|X||\mathcal{F}\right] > M\} \le \mathbb{E}\{\mathbb{E}\left[|X||\mathcal{F}\right]\}/M = \mathbb{E}|X|/M \le \delta.$$

So, by the choice of δ, we have

$$\mathbb{E}(|\mathbb{E}[X|\mathcal{F}]|; |\mathbb{E}[X|\mathcal{F}]| > M) \le \varepsilon \qquad \text{for all } \mathcal{F}.$$

Since ε was arbitrary, the collection is uniformly integrable. □

A common way to check uniform integrability is to use:

Theorem ([1, Theorem 4.6.2])**.** *Let $\varphi \ge 0$ be any function with $\varphi(x)/x \to \infty$ as $x \to \infty$, e.g., $\varphi(x) = x^p$ with $p > 1$ or $\varphi(x) = x \log^+ x$. If $\mathbb{E}\varphi(|X_i|) \le C$ for all $i \in I$, then $\{X_i : i \in I\}$ is uniformly integrable.*

Proof. Let $\varepsilon_M = \sup\{x/\varphi(x) : x \ge M\}$. For $i \in I$,

$$\mathbb{E}\left[|X_i|; |X_i| > M\right] \le \varepsilon_M \mathbb{E}\left[\varphi(|X_i|); |X_i| > M\right] \le C\varepsilon_M$$

and $\varepsilon_M \to 0$ as $M \to \infty$. □

The relevance of uniform integrability to convergence in L^1 is explained by:

Theorem ([1, Theorem 4.6.3]). *Suppose that $\mathbb{E}|X_n| < \infty$ for all n. If $X_n \to X$ in probability then the following are equivalent:*

(i) $\{X_n : n \geq 0\}$ *is uniformly integrable.*

(ii) $X_n \to X$ *in* L^1.

(iii) $\mathbb{E}|X_n| \to \mathbb{E}|X| < \infty$.

Proof. (i) $\Rightarrow$ (ii). Let

$$\phi_M(x) = \begin{cases} M & \text{if } x \geq M, \\ x & \text{if } |x| \leq M, \\ -M & \text{if } x \leq -M. \end{cases}$$

The triangle inequality implies

$$|X_n - X| \leq |X_n - \phi_M(X_n)| + |\phi_M(X_n) - \phi_M(X)| + |\phi_M(X) - X|.$$

Since $|\phi_M(Y) - Y| = (|Y| - M)^+ \leq |Y| 1(|Y| > M)$, taking expected value gives

$$\mathbb{E}|X_n - X| \leq \mathbb{E}|\phi_M(X_n) - \phi_M(X)| + \mathbb{E}\left[|X_n|; |X_n| > M\right] + \mathbb{E}\left[|X|; |X| > M\right].$$

[1, Theorem 2.3.4] implies that $\phi_M(X_n) \to \phi_M(X)$ in probability as $M \to \infty$, so the first term $\to 0$ by the bounded convergence theorem [1, Exercise 2.3.5]. If $\varepsilon > 0$ and M is large, uniform integrability implies that the second term $\leq \varepsilon$. To bound the third term, we observe that uniform integrability implies $\sup \mathbb{E}|X_n| < \infty$, so Fatou's lemma [1, Exercise 2.3.4] implies

$$\mathbb{E}|X| \leq \liminf_{n\to\infty} \mathbb{E}|X_n| < \infty,$$

and by making M larger we can make the third term $\leq \varepsilon$. Combining the last three facts shows $\limsup \mathbb{E}|X_n - X| \leq 2\varepsilon$. Since ε is arbitrary, this proves (ii).

(ii) $\Rightarrow$ (iii). Jensen's inequality implies

$$|\mathbb{E}|X_n| - \mathbb{E}|X|| \leq \mathbb{E}\left||X_n| - |X|\right| \leq \mathbb{E}|X_n - X| \to 0.$$

(iii) $\Rightarrow$ (i). Let

$$\psi_M(x) = \begin{cases} x & \text{on } [0, M-1], \\ 0 & \text{on } [M, \infty), \\ \text{linear on } [M-1, M]. \end{cases}$$

Then $\psi_M(x) \to x$ as $M \to \infty$, so the dominated convergence theorem implies that if M is large, $\mathbb{E}|X| - \mathbb{E}\psi_M(|X|) \leq \varepsilon/2$. As in the first part of the proof, the bounded convergence theorem implies $\mathbb{E}\psi_M(|X_n|) \to \mathbb{E}\psi_M(|X|)$, so using (iii) we get that if $n \geq n_0$,

$$\mathbb{E}\left[|X_n|; |X_n| > M\right] \leq \mathbb{E}|X_n| - \mathbb{E}\psi_M(|X_n|) \leq \mathbb{E}|X| - \mathbb{E}\psi_M(|X|) + \varepsilon/2 < \varepsilon.$$

By choosing M larger, we can make $\mathbb{E}\left[|X_n|; |X_n| > M\right] \leq \varepsilon$ for $0 \leq n < n_0$, so X_n is uniformly integrable. $\square$

We are now ready to state the main theorems of this section. We have already done all the work, so the proofs are short.

Theorem ([1, Theorem 4.6.4]). *For a submartingale, the following are equivalent:*

(i) It is uniformly integrable.

(ii) It converges a.s. and in L^1.

(iii) It converges in L^1.

Proof. (i) $\Rightarrow$ (ii). Uniform integrability implies $\sup \mathbb{E}|X_n| < \infty$ so the martingale convergence theorem implies $X_n \to X$ a.s., and [1, Theorem 4.6.3] implies $X_n \to X$ in L^1.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). $X_n \to X$ in L^1 implies $X_n \to X$ in probability [1, Lemma 2.2.2], so this follows from [1, Theorem 4.6.3]. □

Before proving the analogue of [1, Theorem 4.6.4] for martingales, we will isolate two parts of the argument that will be useful later.

Lemma ([1, Lemma 4.6.5])**.** *If integrable random variables $X_n \to X$ in L^1 then*

$$\mathbb{E}[X_n; A] \to \mathbb{E}[X; A].$$

Proof.

$$|\mathbb{E}X_m 1_A - \mathbb{E}X 1_A| \leq \mathbb{E}\,|X_m 1_A - X 1_A| \leq \mathbb{E}|X_m - X| \to 0.$$

□

Lemma ([1, Lemma 4.6.6])**.** *If a martingale $X_n \to X$ in L^1 then $X_n = \mathbb{E}[X|\mathcal{F}_n]$.*

Proof. The martingale property implies that if $m > n$, $\mathbb{E}[X_m|\mathcal{F}_n] = X_n$, so if $A \in \mathcal{F}_n$, $\mathbb{E}[X_n; A] = \mathbb{E}[X_m; A]$. [1, Lemma 4.6.5] implies $\mathbb{E}[X_m; A] \to \mathbb{E}[X; A]$, so we have $\mathbb{E}[X_n; A] = \mathbb{E}[X; A]$ for all $A \in \mathcal{F}_n$. Recalling the definition of conditional expectation, it follows that $X_n = \mathbb{E}[X|\mathcal{F}_n]$. □

Theorem ([1, Theorem 4.6.7])**.** *For a martingale, the following are equivalent:*

(i) It is uniformly integrable.

(ii) It converges a.s. and in L^1.

(iii) It converges in L^1.

(iv) There is an integrable random variable X so that $X_n = \mathbb{E}(X|\mathcal{F}_n)$.

Proof. (i) $\Rightarrow$ (ii). Since martingales are also submartingales, this follows from [1, Theorem 4.6.4].

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (iv). Follows from [1, Lemma 4.6.6].

(iv) $\Rightarrow$ (i). This follows from [1, Theorem 4.6.1]. □

The next result is related to [1, Lemma 4.6.6] but goes in the other direction.

Theorem ([1, Theorem 4.6.8])**.** *Lévy's 'Upward' Theorem. Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, i.e., $\mathcal{F}_n$ is an increasing sequence of σ-fields and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. As $n \to \infty$,*

$$\mathbb{E}[X|\mathcal{F}_n] \to \mathbb{E}[X|\mathcal{F}_\infty] \qquad \textit{a.s. and in } L^1.$$

Proof. The first step is to note that if $m > n$ then [1, Theorem 4.1.13] implies

$$\mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_m]|\mathcal{F}_n\right] = \mathbb{E}[X|\mathcal{F}_n],$$

so $Y_n = \mathbb{E}[X|\mathcal{F}_n]$ is a martingale. [1, Theorem 4.6.1] implies that Y_n is uniformly integrable, so [1, Theorem 4.6.7] implies that Y_n converges a.s. and in L^1 to a limit Y_∞. The definition of Y_n and [1, Lemma 4.6.6] imply

$$\mathbb{E}[X|\mathcal{F}_n] = Y_n = \mathbb{E}[Y_\infty|\mathcal{F}_n],$$

and hence

$$\int_A X dP = \int_A Y_\infty dP \qquad \text{for all } A \in \mathcal{F}_n.$$

Since X and Y_∞ are integrable, and $\cup_n \mathcal{F}_n$ is a π-system, the π–λ theorem implies that the last result holds for all $A \in \mathcal{F}_\infty$. Since $Y_\infty \in \mathcal{F}_\infty$, it follows that $Y_\infty = \mathbb{E}[X|\mathcal{F}_\infty]$. ☐

Theorem ([1, Theorem 4.6.9])**.** *Lévy's 0–1 law. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$ then*

$$\mathbb{E}(1_A|\mathcal{F}_n) \to 1_A \quad a.s.$$

From Chung: "The reader is urged to ponder over the meaning of this result and judge for himself whether it is obvious or incredible." We will now argue for the two points of view.

"It is obvious." $1_A \in \mathcal{F}_\infty$, and $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, so our best guess of 1_A given the information in $\mathcal{F}_n$ should approach 1_A (the best guess given $\mathcal{F}_\infty$).

"It is incredible." Let $X_1, X_2, \dots$ be independent and suppose $A \in \mathcal{T}$, the tail σ-field. For each n, A is independent of $\mathcal{F}_n$, so $\mathbb{E}(1_A|\mathcal{F}_n) = P(A)$. As $n \to \infty$, the left-hand side converges to 1_A a.s., so $P(A) = 1_A$ a.s., and it follows that $P(A) \in \{0, 1\}$, i.e., we have proved Kolmogorov's 0–1 law.

References

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[2] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.