Homotopy Reconstruction via the Cech Complex and the Vietoris-Rips Complex

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Paper

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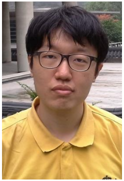
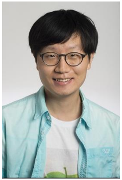


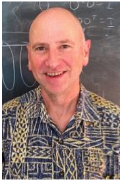
Jisu Kim



Jaehyeok Shin

Frédéric Chazal

Alessandro Rinaldo



Larry Wasserman

Introduction

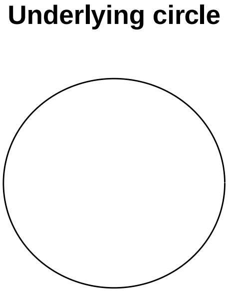
The nerve theorem for Euclidean sets of positive reach

Homotopy Equivalence on positive μ-reach

Homotopy Reconstruction via Cech complex and Vietoris-Rips complex

We are interested in estimating the topology of the target space $\mathbb{X} \subset \mathbb{R}^d$ based on samples $\mathcal{X}$ that lies in it or in its proximity.

- Estimating the topology from samples occurs in: cosmology, time series data, machine learning, etc.



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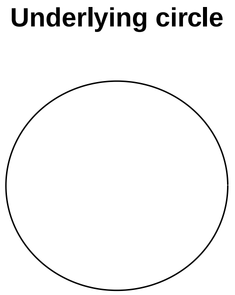
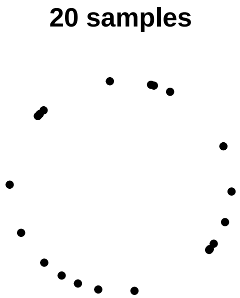
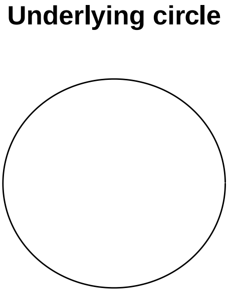


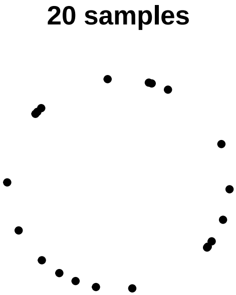


Two topological spaces are homotopy equivalent if they can be continuously deformed into one another.

- We estimate topology up to homotopy equivalence.
- A space X is contractible if X is homotopy equivalent to a point.

We estimate the homotopy of the target space $\mathbb{X} \subset \mathbb{R}^d$ based on samples $\mathcal{X}$ that lies in it or in its proximity.

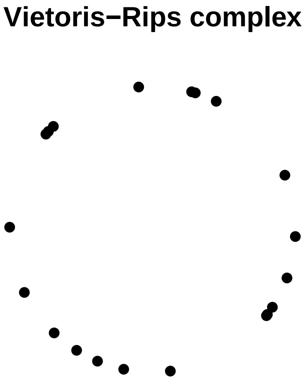




We use the Vietoris-Rips complex to estimate the target space.

- For $\mathcal{X} \subset \mathbb{R}^d$ and $r > 0$, the Vietoris-Rips complex $\mathrm{Rips}(\mathcal{X}, r)$ is defined as

$$\mathrm{Rips}(\mathcal{X}, r) = \{\{x_1, \ldots, x_k\} \subset \mathcal{X} : d(x_i, x_j) < 2r, \text{ for all } 1 \leq i, j \leq k\}.$$



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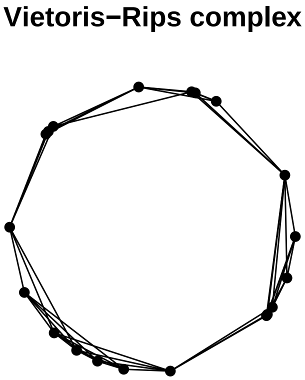

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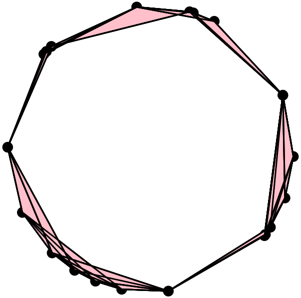
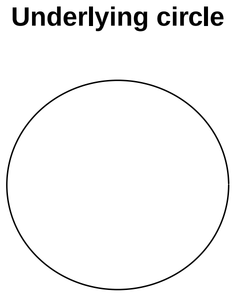
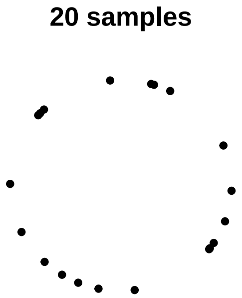
Vietoris–Rips complex



We estimate the homotopy of the target space $\mathbb{X} \subset \mathbb{R}^d$ using the Vietoris-Rips complex on samples $\mathcal{X}$ that lies in it or in its proximity.





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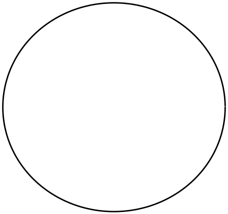
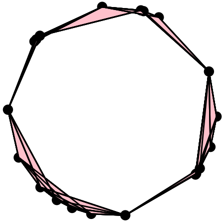
Underlying circle

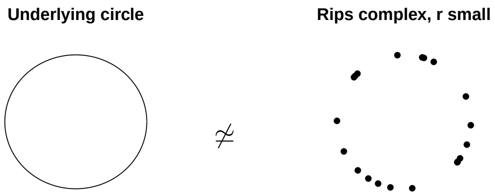


Vietoris−Rips complex



The homotopy of the Vietoris-Rips complex on samples $\mathcal{X}$ can be very different from the homotopy of the target space $\mathbb{X} \subset \mathbb{R}^d$.

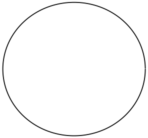
- If r is too small, then $\mathrm{Rips}(\mathcal{X}, r)$ is a set of disconnected points



The homotopy of the Vietoris-Rips complex on samples $\mathcal{X}$ can be very different from the homotopy of the target space $\mathbb{X} \subset \mathbb{R}^d$.

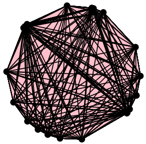
- If r is too large, then $\mathrm{Rips}(\mathcal{X}, r)$ is homotopy equivalent to a point.

Underlying circle



$\not\simeq$

Rips complex, r large



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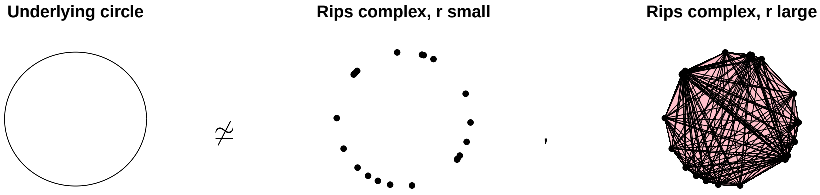

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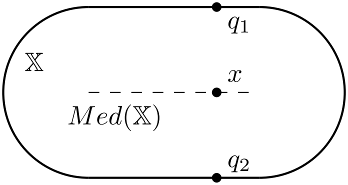
- The medial axis is the set of points that have at least two nearest neighbors, i.e.

$$Med(\mathbb{X}) = \{x \in \mathbb{R}^d : \text{ there exists } q_1 \neq q_2 \in \mathbb{X} \text{ with } \|q_1 - x\| = \|q_2 - x\| = d(x, \mathbb{X})\}.$$



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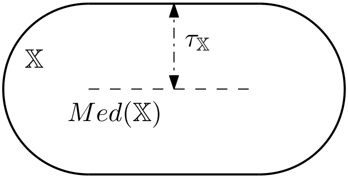
$$\tau_{\mathbb{X}} = \inf_{q \in \mathbb{X}} d(q, Med(\mathbb{X})) = \inf_{q \in \mathbb{X}, x \in Med(\mathbb{X})} \|q - x\|.$$

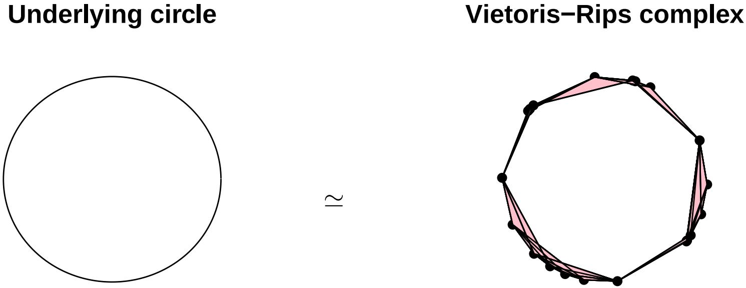


We derive conditions under which the homotopy of the target space $\mathbb{X} \subset \mathbb{R}^d$ is correctly reconstructed using the Vietoris-Rips complex on samples $\mathcal{X}$.

- Kim et al. [2020]: extending Niyogi et al. [2008], Attali et al. [2013]

Theorem

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with its reach $\tau > 0$, and let $\mathcal{X} \subset \mathbb{R}^d$ be a closed discrete set of points. If $\frac{d_H(\mathbb{X},\mathcal{X})}{\tau} \leq C$ where $d_H(\mathbb{X},\mathcal{X})$ is the Hausdorff distance, then with appropriate choice of r, the Vietoris-Rips complex $Rips(\mathcal{X}, r)$ is homotopy equivalent to $\mathbb{X}$, with $C \approx 0.07856 \cdots$.



Introduction

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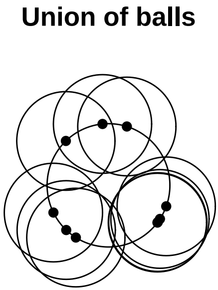
We aproximate the target space by the union of balls centered at data points.

- Let $r > 0$ be a pre-specified radius, then we approximate the target space $\mathbb{X}$ by the union of restricted balls centered at data points

$$\bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{X}}(x, r),$$

where $\mathbb{B}_{\mathbb{X}}(x, r) = \{y \in \mathbb{X} : \|y - x\| < r\}$ is the restricted ball of radius r centered at x.



Čech complex is constructed by finding nonempty intersections of balls.

- For $\mathbb{X}, \mathcal{X} \subset \mathbb{R}^d$ and $r > 0$, the restricted Čech complex $\check{C}ech_{\mathbb{X}}(\mathcal{X}, r)$ is defined as

$$\check{C}ech_{\mathbb{X}}(\mathcal{X}, r) = \left\{ \{x_1, \ldots, x_k\} \subset \mathcal{X} : \bigcap_{j=1}^{k} \mathbb{B}_{\mathbb{X}}(x_j, r) \neq \emptyset \right\},$$

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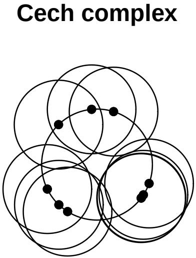

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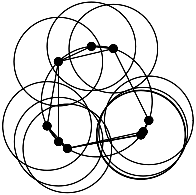
Cech complex



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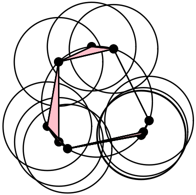
Cech complex



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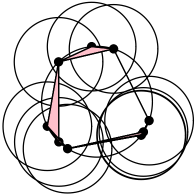
  where $\mathbb{B}_{\mathbb{X}}(x, r) = \{y \in \mathbb{X} : \|y - x\| < r\}$ is the restricted ball of radius r centered at x.
- The ambient Čech complex $\check{C}ech_{\mathbb{R}^d}(\mathcal{X}, r)$ is when $\mathbb{X} = \mathbb{R}^d$.

Cech complex

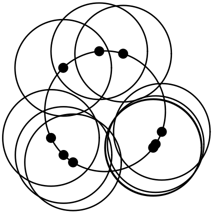


The union of balls and the restricted Čech complex are homotopy equivalent if any non-empty intersection of restricted balls is contractible.

Theorem (Nerve Theorem)

If any nonempty intersection of finitely many sets in $\{\mathbb{B}_{\mathbb{X}}(x, r_x) : x \in \mathcal{X}\}$ *is contractible, then the union of balls* $\bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{X}}(x, r)$ *and the restricted Čech complex* $\check{C}ech_{\mathbb{X}}(\mathcal{X}, r)$ *are homotopy equivalent.*

Union of balls | **Restricted Cech complex**



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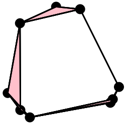
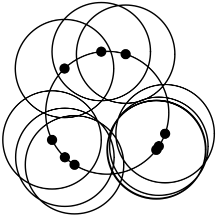


For a positive reach set, any non-empty intersection of restricted balls is contractible.

Theorem (Kim et al. [2020, Theorem 9])

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with its reach $\tau > 0$, $\mathcal{X} \subset \mathbb{R}^d$ be a set of points, and $r > 0$. Then, if $r \leq \sqrt{\tau^2 + (\tau - d_{\mathbb{X}}(x))^2}$ for all $x \in \mathcal{X}$, then $\bigcap_{x \in I} \mathbb{B}_{\mathbb{X}}(x, r)$ for $I \subset \mathcal{X}$ is contractible.



The homotopy of a positive reach set is reconstructed from the restricted Čech complex.

Corollary (Kim et al. [2020, Corollary 10])

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with its reach $\tau > 0$, $\mathcal{X} \subset \mathbb{R}^d$ be a set of points, and $r > 0$. Then, if $r \leq \sqrt{\tau^2 + (\tau - d_{\mathbb{X}}(x))^2}$ for all $x \in \mathcal{X}$, then $\bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{X}}(x, r)$ is homotopy equivalent to the restricted Čech complex $\check{C}ech_{\mathbb{X}}(\mathcal{X}, r)$. In addition, if $\mathbb{X} \subset \bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{X}}(x, r)$, then $\mathbb{X}$ is homotopy equivalent to $\check{C}ech_{\mathbb{X}}(\mathcal{X}, r)$.

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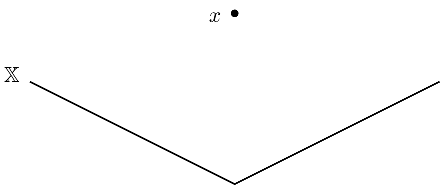
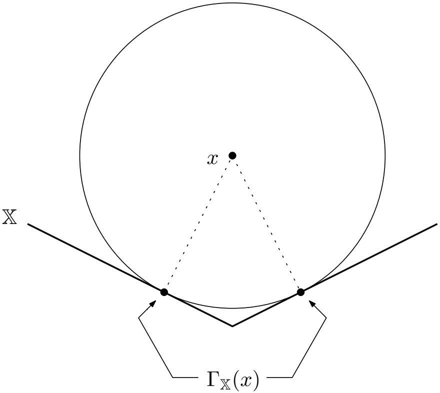
The generalized gradient of the distance function

- For $x \in \mathbb{R}^d \backslash \mathbb{X}$,



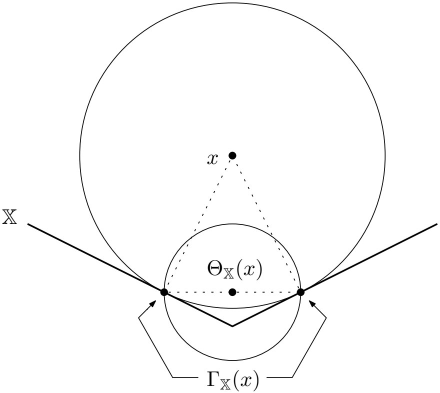
The generalized gradient of the distance function

- For $x \in \mathbb{R}^d \backslash \mathbb{X}$, let $\Gamma_{\mathbb{X}}(x)$ be the set of points in $\mathbb{X}$ closest to x,



The generalized gradient of the distance function

- For $x \in \mathbb{R}^d \backslash \mathbb{X}$, let $\Gamma_{\mathbb{X}}(x)$ be the set of points in $\mathbb{X}$ closest to x, $\Theta_{\mathbb{X}}(x)$ be the center of the unique smallest ball enclosing $\Gamma_{\mathbb{X}}(x)$,



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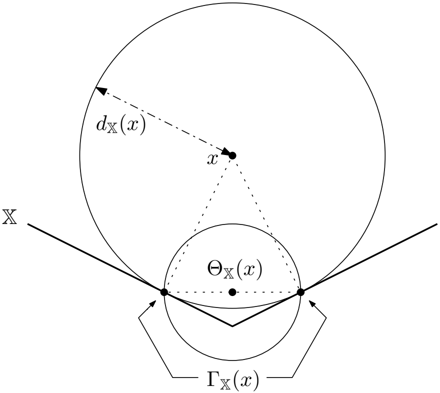
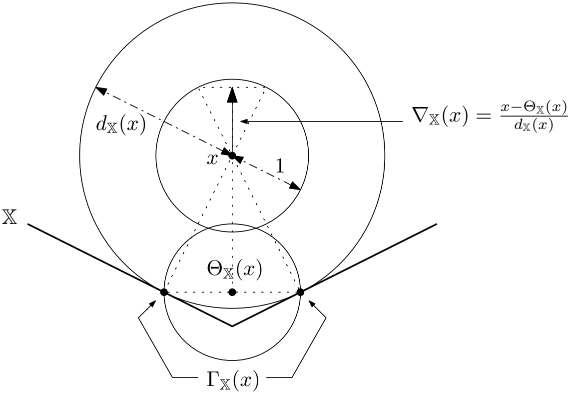


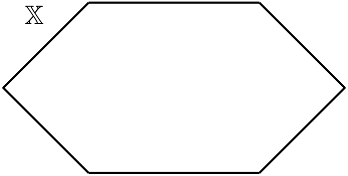
The generalized gradient of the distance function

- For $x \in \mathbb{R}^d \backslash \mathbb{X}$, let $\Gamma_{\mathbb{X}}(x)$ be the set of points in $\mathbb{X}$ closest to x, $\Theta_{\mathbb{X}}(x)$ be the center of the unique smallest ball enclosing $\Gamma_{\mathbb{X}}(x)$, and let $d_{\mathbb{X}}(x) = d(x, \mathbb{X})$ be the distance from x to $\mathbb{X}$. The generalized gradient of the distance function, denoted by $\nabla_{\mathbb{X}}(x)$, is defined as

$$\nabla_{\mathbb{X}}(x) = \frac{x - \Theta_{\mathbb{X}}(x)}{d_{\mathbb{X}}(x)}.$$



The μ-reach of $\mathbb{X}$, denoted by $\tau_{\mathbb{X}}^{\mu}$, is the minimum distance from the μ-medial axis to the set.



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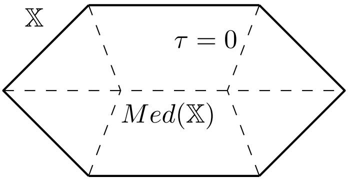

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- The μ-medial axis is the set of points defined as

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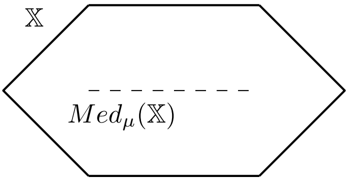

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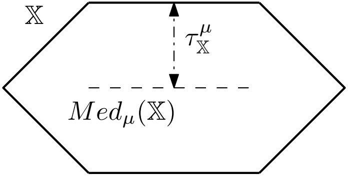
- The μ-medial axis is the set of points defined as

$$Med_{\mu}(\mathbb{X}) = \{x \in \mathbb{R}^d \backslash \mathbb{X} : \ \|\nabla_{\mathbb{X}}(x)\| < \mu\}.$$

- The μ-reach is the minimum distance from the μ-medial axis to the set, i.e.

$$\tau_{\mathbb{X}}^{\mu} = \inf_{q \in \mathbb{X}} d(q, Med_{\mu}(\mathbb{X})) = \inf_{q \in \mathbb{X}, x \in Med_{\mu}(\mathbb{X})} \|q - x\|.$$



For a positive μ-reach set, it and its offset are homotopy equivalent.

- For a set $\mathbb{X} \subset \mathbb{R}^d$ and $r > 0$, its r-offset $\mathbb{X}^r$ is $\mathbb{X}^r := \{x \in \mathbb{R}^d : d(x, \mathbb{X}) < r\}$.

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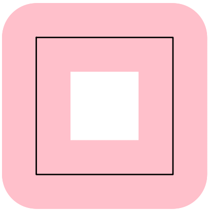

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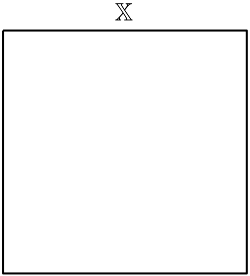
Theorem (Kim et al. [2020, Theorem 12])

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with positive μ-reach $\tau^\mu > 0$. For $r \leq \tau^\mu$, $\mathbb{X}$ and $\mathbb{X}^r$ are homotopy equivalent.

For a positive μ-reach set, its double offset is homotopy equivalent to the set and has positive reach.

- For $s, t > 0$ with $t \leq s$, $\mathbb{X}^{s,t} := (((\mathbb{X}^s)^{\complement})^t)^{\complement}$ is the double offset of $\mathbb{X}$.



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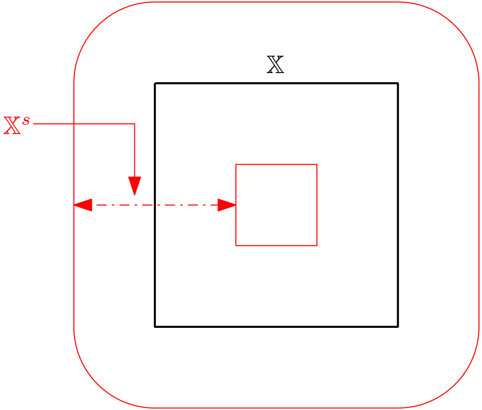

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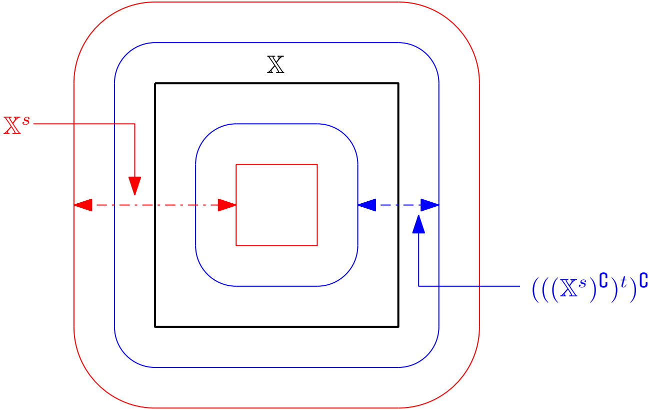

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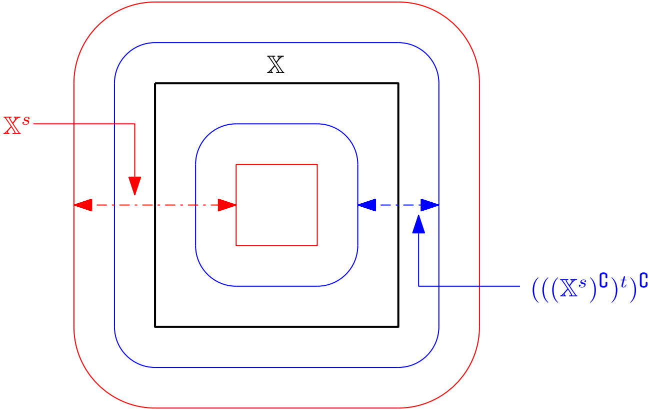
Corollary (Kim et al. [2020, Corollary 15])

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with positive μ-reach $\tau^\mu > 0$. For $s, t > 0$ with $t \leq s$, let $\mathbb{X}^{s,t} := (((\mathbb{X}^s)^\complement)^t)^\complement$ be the double offset of $\mathbb{X}$. If $s < \tau^\mu$ and $t < \mu s$, then $\mathbb{X}^{s,t}$ and $\mathbb{X}$ are homotopy equivalent, and $\tau_{\mathbb{X}^{s,t}} \geq t$.



Introduction

The nerve theorem for Euclidean sets of positive reach

Homotopy Equivalence on positive μ-reach

Homotopy Reconstruction via Cech complex and Vietoris-Rips complex

The homotopy of a positive reach set is reconstructed from the ambient Čech complex.

Theorem

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with reach $\tau > 0$ and $\mathcal{X} \subset \mathbb{R}^d$ be a closed discrete set of points. Let $r > 0$, $\epsilon := \max_{x \in \mathcal{X}}\{d_{\mathbb{X}}(x)\}$, and $\delta > 0$ be satisfying

$$\mathbb{X} \subset \bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{R}}(x, \delta) \qquad \textit{and} \qquad r \leq \tau - \epsilon.$$

Then with sufficiently small δ, the ambient Čech complex $\check{C}ech_{\mathbb{R}^d}(\mathcal{X}, r)$ is homotopy equivalent to $\mathbb{X}$.

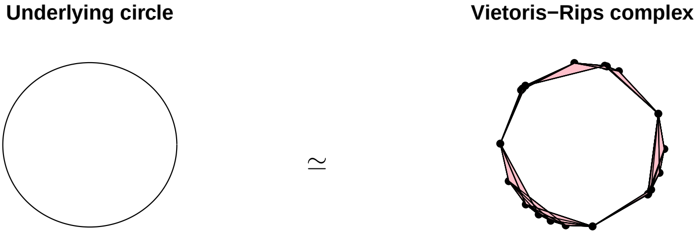
The homotopy of a positive reach set is reconstructed from the Vietoris-Rips complex.

Theorem

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with reach $\tau > 0$ and $\mathcal{X} \subset \mathbb{R}^d$ be a closed discrete set of points. Let $r > 0$, $\epsilon := \max_{x \in \mathcal{X}} \{d_{\mathbb{X}}(x)\}$, and $\delta > 0$ be satisfying

$$\mathbb{X} \subset \bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{R}}(x, \delta) \qquad \text{and} \qquad r \leq \sqrt{\frac{d+1}{2d}}(\tau - \epsilon).$$

Then with sufficiently small δ, the Vietoris-Rips complex $Rips(\mathcal{X}, r)$ is homotopy equivalent to $\mathbb{X}$.



The homotopy of a positive μ-reach set is reconstructed from the ambient Čech complex and the Vietoris-Rips complex.

Corollary

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with μ-reach $\tau^\mu > 0$ and $\mathcal{X} \subset \mathbb{R}^d$ be a closed discrete set of points. Let $r > 0$. For $s \geq t \geq 0$ with $\frac{t}{\mu} < s < \tau^\mu$, let $\mathbb{Y} := (((\mathbb{X}^s)^{\complement})^t)^{\complement}$ be a double offset of $\mathbb{X}$. Let $\epsilon := \max_{x \in \mathcal{X}} \{d_{\mathbb{Y}}(x)\}$ and $\delta > 0$ be satisfying

$$\mathbb{Y} \subset \bigcup_{x \in \mathcal{X}} \mathbb{B}_{\mathbb{R}}(x, \delta).$$

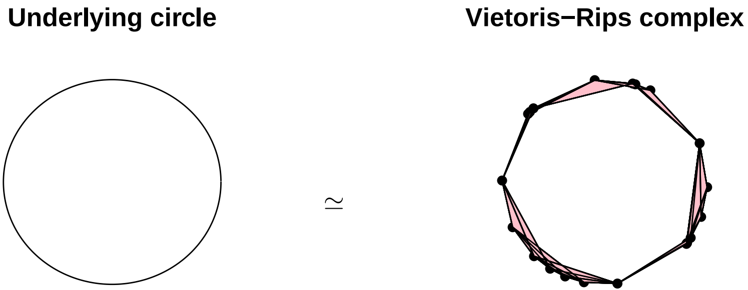
Then with appropriate condition on r and sufficiently small δ, the ambient Čech complex $\check{C}ech_{\mathbb{R}^d}(\mathcal{X}, r)$ and the Vietoris-Rips complex $Rips(\mathcal{X}, r)$ are homotopy equivalent to $\mathbb{X}$.

The homotopy of a positive reach set can be reconstructed from the Vietoris-Rips complex.

Corollary

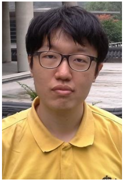
Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with its reach $\tau > 0$, and let $\mathcal{X} \subset \mathbb{R}^d$ be a closed discrete set of points. If $\frac{d_H(\mathbb{X},\mathcal{X})}{\tau} \leq C$ where $d_H(\mathbb{X},\mathcal{X})$ is the Hausdorff distance, then with appropriate choice of r, the Vietoris-Rips complex $Rips(\mathcal{X}, r)$ is homotopy equivalent to $\mathbb{X}$, with $C \approx 0.07856\cdots$.

- Previous result: $C \approx 0.03412$ in Attali et al. [2013]

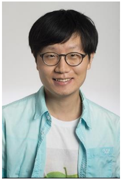


Thank you!

- Jisu Kim, Jaehyeok Shin, Frédéric Chazal, Alessandro Rinaldo, Larry Wasserman, Homotopy Reconstruction via the Cech Complex and the Vietoris-Rips Complex
- DOI: https://doi.org/10.4230/LIPIcs.SoCG.2020.54
- arXiv: https://arxiv.org/abs/1903.06955
- HAL: https://hal.archives-ouvertes.fr/hal-02425686



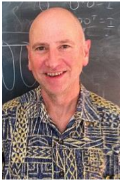
Jisu Kim



Jaehyeok Shin

Frédéric Chazal

Alessandro Rinaldo



Larry Wasserman

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Jisu Kim, Jaehyeok Shin, Frédéric Chazal, Alessandro Rinaldo, and Larry Wasserman. Homotopy Reconstruction via the Cech Complex and the Vietoris-Rips Complex. In Sergio Cabello and Danny Z. Chen, editors, *36th International Symposium on Computational Geometry (SoCG 2020)*, volume 164 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 54:1–54:19, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. ISBN 978-3-95977-143-6. doi: 10.4230/LIPIcs.SoCG.2020.54. URL `https://drops.dagstuhl.de/opus/volltexte/2020/12212`.

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Introduction

The nerve theorem for Euclidean sets of positive reach

Deformation retraction on positive μ-reach

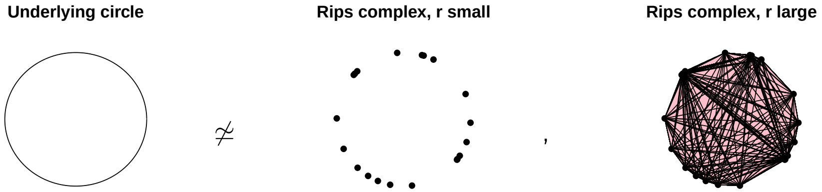
Two topological spaces are homotopy equivalent if they can be continuously deformed into one another.

- Two functions $f, g : X \to Y$ are homotopic if there exists a continuous funciton $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We write $f \simeq g$.
- Two spaces X, Y are homotopy equivalent if there exists continuous $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We write $X \simeq Y$.

The topology of the Vietoris-Rips complex on samples $\mathcal{X}$ can be very different from the topology of the target space $\mathbb{X} \subset \mathbb{R}^d$.

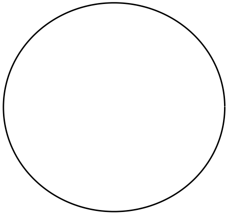
- From Adamaszek and Adams [2017], if $\frac{\sqrt{3}}{2} < r < 1$, then

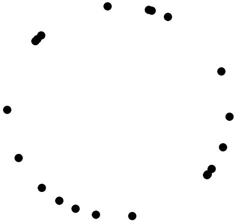
$$\mathrm{Rips}(\mathcal{X}, r) \simeq S^{2l+1} \text{ for some } l \geq 1, \text{ or}$$
$$\mathrm{Rips}(\mathcal{X}, r) \simeq \vee^c S^{2l} \text{ for some } l \geq 0 \text{ and } c \geq 0.$$



Hausdorff distance measures the distance between two subsets.

- For two subsets $X, Y \subset \mathbb{R}^d$, the Hausdorff distance between X and Y is defined as $d_H(X, Y) := \inf\{r > 0 : X \subset Y^r \text{ and } Y \subset X^r\}$, where $X^r = \{x \in \mathbb{R}^d : d(x, X) < r\}$ is the r-offset of X.





Introduction

The nerve theorem for Euclidean sets of positive reach

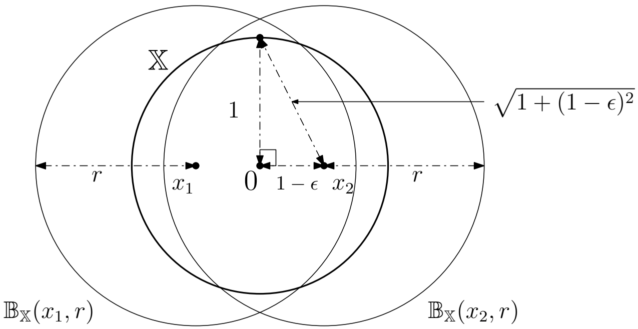
Deformation retraction on positive μ-reach

The reach condition on radius $r \leq \sqrt{\tau^2 + (\tau - d_{\mathbb{X}}(x))^2}$ is tight.

Example

Let $\mathbb{X} = S^1 \subset \mathbb{R}^2$, fix $\epsilon > 0$, $x_1 = (1-\epsilon, 0)$, $x_2 = (-1+\epsilon, 0)$, $\mathcal{X} = \{x_1, x_2\}$. Then if $r > \sqrt{1 + (1-\epsilon)^2}$, then

$$\mathbb{B}_{\mathbb{X}}(x_1, r) \bigcup \mathbb{B}_{\mathbb{X}}(x_2, r) \simeq \mathbb{X}, \qquad \text{but} \qquad \check{C}ech_{\mathbb{X}}(\mathcal{X}, r) \simeq 0.$$



Introduction

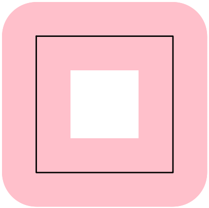
The nerve theorem for Euclidean sets of positive reach

Deformation retraction on positive μ-reach

For a subset with nonvanishing generalized gradient, its offsets are homotopy equivalent.

- For a set $\mathbb{X} \subset \mathbb{R}^d$ and $r > 0$, its r-offset $\mathbb{X}^r$ is $\mathbb{X}^r := \{x \in \mathbb{R}^d : d(x, \mathbb{X}) < r\}$.

0.5–offset of square



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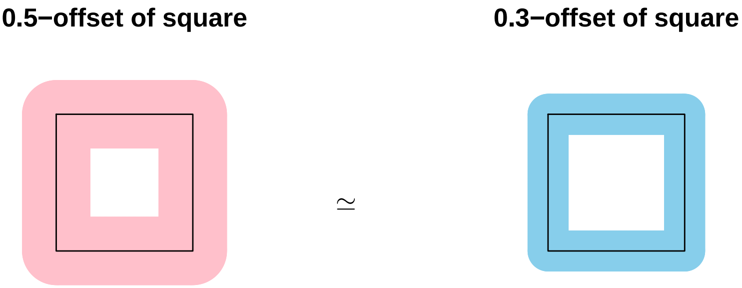
Lemma (Isotopy Lemma)

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset, and for $r, s > 0$ with $s \leq r$, let $\mathbb{X}^s, \mathbb{X}^r$ be two offsets of $\mathbb{X}$. Suppose $\nabla_{\mathbb{X}}(x) \neq 0$ for all $x \in \overline{\mathbb{X}^r} \backslash \mathbb{X}^s$. Then $\mathbb{X}^r$ and $\mathbb{X}^s$ are homeomorphic, and hence homotopy equivalent.



For a positive μ-reach set, its offset deformation retracts to itself.

Theorem (Kim et al. [2020, Theorem 12])

Let $\mathbb{X} \subset \mathbb{R}^d$ be a subset with positive μ-reach $\tau^{\mu} > 0$. For $r \leq \tau^{\mu}$, the r-offset $\mathbb{X}^r$ deformation retracts to $\mathbb{X}$. In particular, $\mathbb{X}$ and $\mathbb{X}^r$ are homotopy equivalent.

The positive μ-reach condition $r \leq \tau^{\mu}$ is critical.

Example

Let $\mathbb{X}$ be a topologist's sine circle, i.e., $\mathbb{X} = \mathbb{X}_0 \cup \mathbb{X}_1 \cup \mathbb{X}_2$, $\mathbb{X}_0 = \{(x, \sin \frac{1}{x}) : x \in (0, 1]\}$, $\mathbb{X}_1 = \{0\} \times [-1, 1]$, and $\mathbb{X}_2$ is a curve joining $(0, 1)$ and $(1, 0)$. Then, $\tau^{\mu}_{\mathbb{X}} = 0$ for any $\mu \in (0, 1]$, but $\nabla_{\mathbb{X}}$ is nonzero for all $x \in \mathbb{R}^2 \backslash \mathbb{X}$. Now, for sufficiently small $r > 0$,

$$H_1(\mathbb{X}) = 0, \qquad \text{but} \qquad H_1(\mathbb{X}^r) = \mathbb{Z},$$

so $\mathbb{X}^r$ cannot deformation retract to $\mathbb{X}$.