Minimax Rate for Estimating the Dimension of a Manifold

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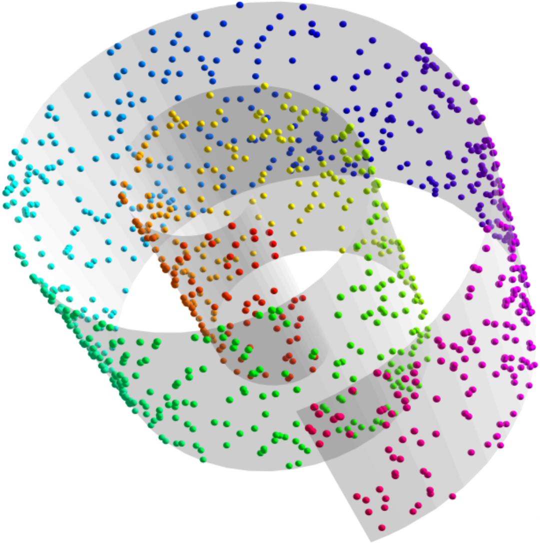
Manifold Learning Finds an Underlying Manifold to Reduce Dimension.



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[1] http://www.skybluetrades.net/blog/posts/2011/10/30/machine-learning/

Intrinsic Dimension of a Manifold needs to be Estimated.

- Most manifold learning algorithms require the intrinsic dimension of the manifold as input.
- Intrinsic dimension is rarely known in advance and therefore has to be estimated.

Minimax Rate is of Interest.

- Minimax rate is the risk of an estimator that performs best in the worst case, as a function of the sample size.
- $$R_n = \inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}} \underbrace{\mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\dim}_n(X),\ \dim(P) \right) \right]}_{\text{risk of an estimator}}$$
  - $X = (X_1, \cdots, X_n)$ is drawn from a fixed distribution P, where P is contained in set of distributions $\mathcal{P}$.
  - estimator $\hat{\dim}_n$ is any function of the data X.

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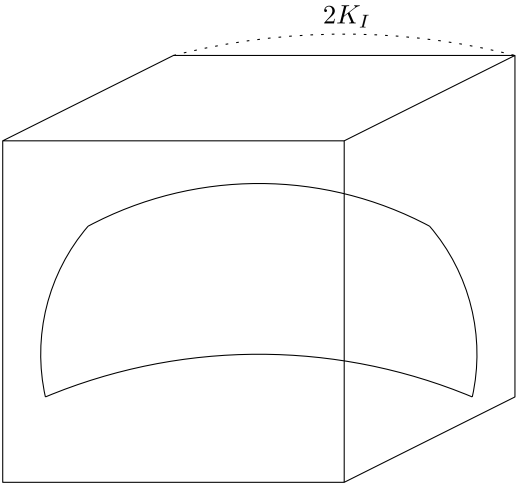
Regularity Conditions on Measures and Conditions on Distributions and Supporting Manifolds are Assumed.



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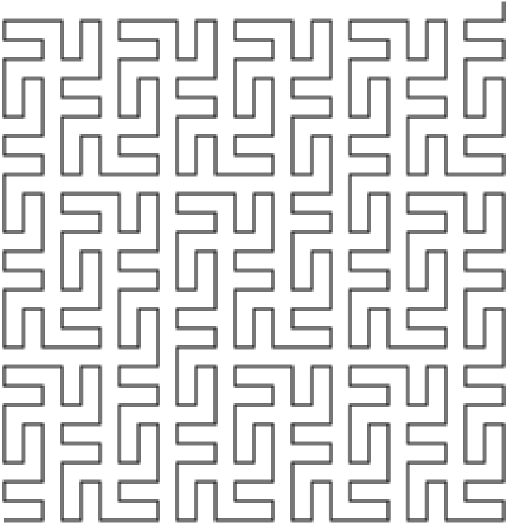
 [2]

[2] https://people.csail.mit.edu/jaffer/Geometry/PSFC

Regularity Conditions on Measures and Conditions on Distributions and Supporting Manifolds are Assumed.

$$\frac{dP}{dvol_M} \leq K_p$$

Regularity Conditions on Measures and Conditions on Distributions and Supporting Manifolds are Assumed.

- $\mathcal{P}^d$ denotes set of distributions P with bounded support, bounded curvature, and bounded density.

Binary Classification and $0 - 1$ loss are Considered.

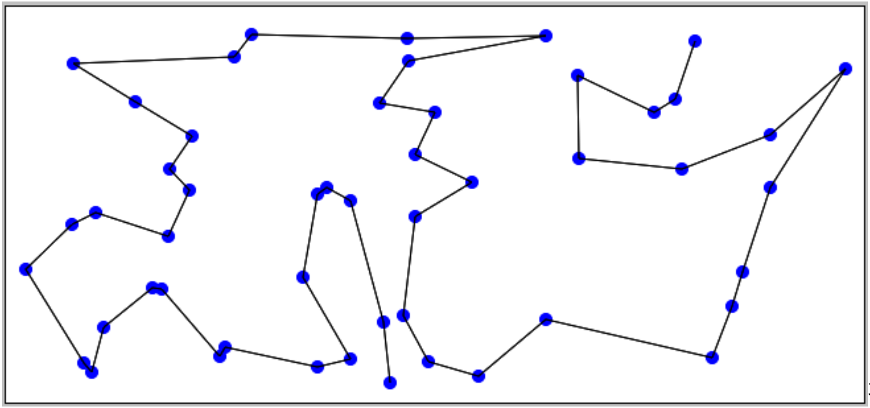
$$R_n = \inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\dim}_n(X),\ \dim(P) \right) \right]$$

- We assume that the manifolds are of two possible dimensions, d_1 and d_2, so the considered distribution set is $\mathcal{P} = \mathcal{P}^{d_1} \cup \mathcal{P}^{d_2}$.
- $0 - 1$ loss function is considered, so for all $x, y \in \mathbb{R}$, $\ell(x, y) = I(x \neq y)$.

The Maximum Risk of any chosen Estimator Provides an Upper Bound on the Minimax Rate.

$$
\begin{aligned}
R_n &= \inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell\left(\hat{\dim}_n(X), \dim(P) \right) \right] \\
&\leq \underbrace{\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell\left(\hat{\dim}_n(X), \dim(P) \right) \right]}_{\text{the maximum risk of any chosen estimator}}
\end{aligned}
$$

TSP(Travelling Salesman Problem) Path Finds Shortest Path that Visits Each Points exactly Once.


[3]

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[3] http://www.heatonresearch.com/fun/tsp/anneal

Our Estimator estimates Dimension to be d_2 if d_1-squared Length of TSP Generated by the Data is Long.

- When intrinsic dimesion is higher, length of TSP path is likely to be longer.
- $$\hat{\dim}_n(X) = d_1 \iff$$
  $$\exists \sigma \in S_n \ s.t \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \leq C,$$

  where C is some constant that depends only on regularity conditions.

Mimimax Rate is Upper Bounded by $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$.

Our estimator has maximum risk of $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$.

Proposition

Let $1 \leq d_1 < d_2 \leq m$. Then

$$\inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}^{d_1} \cup \mathcal{P}^{d_2}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\dim}_n, \dim(P)\right)\right] \lesssim n^{-\left(\frac{d_2}{d_1}-1\right)n}.$$

Le Cam's Lemma provides Lower Bounds based on the Minimum of Two Densities $q_1 \wedge q_2$, where q_i are in Convex Hull of $\mathcal{P}^{d_i}$.

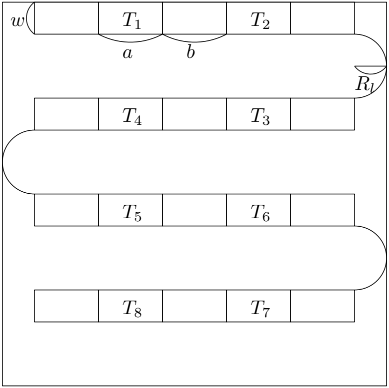
Lemma

Let $\mathcal{P}$ be a set of probability measures, and $\mathcal{P}^{d_1}, \mathcal{P}^{d_2} \subset \mathcal{P}$ be such that for all $P \in \mathcal{P}^{d_i}$, $\theta(P) = \theta_i$ for $i = 1, 2$. For any $Q_i \in co(\mathcal{P}_i)$, let q_i be density of Q_i with respect to measure ν. Then

$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P[d(\hat{\theta}, \theta(P))] \geq \frac{d(\theta_1, \theta_2)}{4} \sup_{Q_i \in co(\mathcal{P}^{d_i})} \int [q_1(x) \wedge q_2(x)] d\nu(x).$$

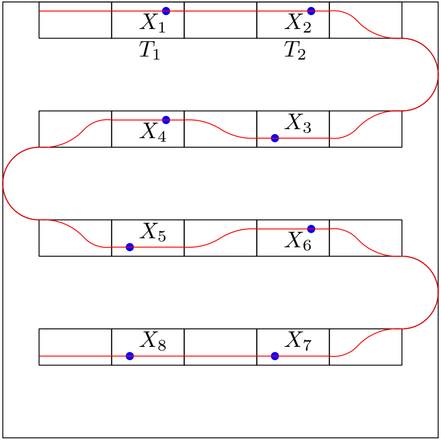
T is Constructed so that for any $x = (x_1, \cdots, x_n) \in T$, there exists a d_1-dimensional Manifold belonging to the Model and Passing through $x_1, \cdots, x_n$.

- T_i's are cylinder sets in $[-K_I, K_I]^{d_2}$, and then T is constructed as $T = S_n \prod_{i=1}^{n} T_i$, where the permutation group S_n acts on $\prod_{i=1}^{n} T_i$ as a coordinate change.



T is Constructed so that for any $x = (x_1, \cdots, x_n) \in T$, there exists a d_1-dimensional Manifold belonging to the Model and Passing through $x_1, \cdots, x_n$.

- Given $x_1, \cdots, x_n \in T$ (blue points), a manifold in the model (red line) passes through $x_1, \cdots, x_n$.



$\mathcal{P}^{d_1}$ is Constructed as Set of Distributions that are Supported on Manifolds that Passes through $x_1, \cdots, x_n$ for $x = (x_1, \cdots, x_n) \in T$, and $\mathcal{P}^{d_2}$ is a Singleton Set Consisting of the Uniform Distirbution on $[-K_I, K_I]^{d_2}$.

Mimimax Rate is Lower Bounded by $\Omega\left(n^{-2(d_2-d_1)n}\right)$.

- The lower bound below is from Le Cam's lemma with the constructed $\mathcal{P}_1^{d_1}$ and $\mathcal{P}_2^{d_2}$.

Proposition

$$\inf_{\hat{\dim}} \sup_{P \in \mathcal{P}^{d_1} \cup \mathcal{P}^{d_2}} \mathbb{E}_{P^{(n)}}[l(\hat{\dim}_n, \dim(P))] \gtrsim n^{-2(d_2-d_1)n}.$$

Multinary Classification and $0 - 1$ Loss are Considered.

- $$R_n = \inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\dim}_n(X),\ \dim(P) \right) \right]$$
- Now the manifolds are of any dimensions between 1 and m, so considered distribution set is $\mathcal{P} = \bigcup_{d=1}^{m} \mathcal{P}^d$.
- $0 - 1$ loss function is considered, so for all $x, y \in \mathbb{R}$, $\ell(x, y) = I(x = y)$.

Mimimax Rate is Upper Bounded by $O\left(n^{-\frac{1}{m-1}n}\right)$, and Lower Bounded by $\Omega\left(n^{-2n}\right)$.

Proposition

$$n^{-2n} \lesssim \inf_{\hat{\dim}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\dim}_n, \dim(P)\right)\right] \lesssim n^{-\frac{1}{m-1}n}.$$

Thank you!