Minimax Reach Estimator

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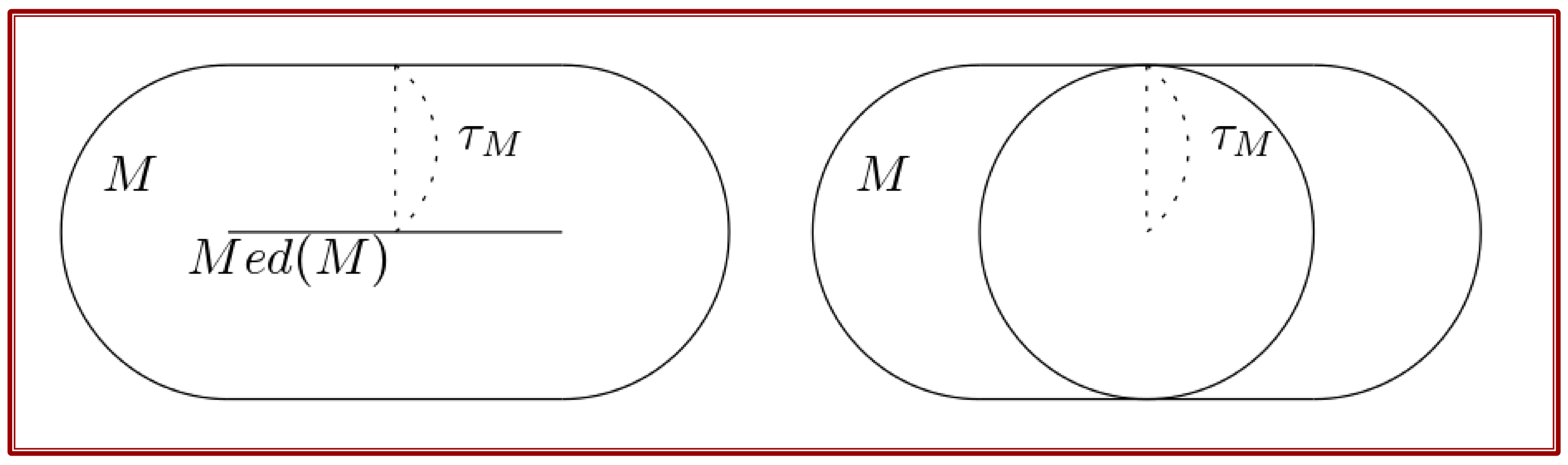
Abstract

Regularity parameters are crucial to derive approximation properties as well as in implementation. The reach has been proven to play a key role in computational geometry. It carries both local and global regularity information, and can be seen as a minimal scale parameter. The goal is to propose an estimator of the reach τ_M of a d-dimensional sub-manifold $M \subset \mathbb{R}^m$ given a random i.i.d. sample $\{X_1, \ldots, X_n\}$. We give minimax bounds for reach estimation over a class of manifolds satisfying natural geometric constraints.

Definition

Reach



- Medial axis of M, $Med(M)$, is the set of points in $\mathbb{R}^m$ that do not have unique nearest neighbors on M.
- The reach of M, denoted by τ_M, is the minimum distance from $Med(M)$ to M: $\tau_M = \inf_{x \in Med(M), y \in M} ||x - y||_2$.
- The reach τ_M gives maximal offset size of M on which the projection is well defined.
- The reach τ_M also gives the maximum radius of a ball that you can roll over M.

Minimax rate

- Suppose you observe iid data X and the set of distributions $\mathcal{P}$ and loss function $l(,)$ is fixed, and you are interested in parameter θ.
- Maximum risk of an estimator $\hat{\theta}$ is the risk that the estimator can make in the worst case, i.e.
$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\theta}(X), \theta(P)\right)\right].$$
- Minimax rate is the infimum of a maximum risk over all possible estimators, $\hat{\theta}$ i.e.
$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\theta}(X), \theta(P)\right)\right].$$
- In our case, loss function is $l(a, b) = \left|\frac{1}{a} - \frac{1}{b}\right|^p$, and the parameter of interest is the reach τ_P.

Regularity conditions

We consider $\mathcal{P}$ to be set of distributions with regular conditions:

- The reach τ_M of the support manifold M is lower bounded by $\tau_{\min}$.
- All the arc-length parametrized geodesic γ on the support manifold M has L-Lipsctiz 2nderivatives: for all $t, s \in \mathbb{R}$, $||\gamma''(t) - \gamma''(s)|| \le L|t - s|$.
- The probability distribution P has density f with respect to uniform measure on M satisfying $f_{min} \le f(x) \le f_{max}$.

Reach estimators

- The reach τ_M of M can be characterized as
$$\frac{1}{\tau_M} = \sup_{a, b \in M, a \ne b} \frac{d(b - a, T_a M)}{||b - a||_2^2}.$$

When tangent spaces are known:

- We consider the plugin estimator
$$\frac{1}{\hat{\tau}} = \sup_{1 \le i \ne j \le n} \frac{d(X_j - X_i, T_{X_i} M)}{||X_j - X_i||_2^2}.$$

When tangent spaces are unknown:

- $\hat{T}_i$ be an estimator of $T_{X_i} M$, then we consider the plugin estimator
$$\frac{1}{\hat{\tau}} = \sup_{1 \le i \ne j \le n} \frac{d(X_j - X_i, \hat{T}_i)}{||X_j - X_i||_2^2}.$$

Upper bound and Lower bound of Minimax Rate

Upper bound

- The maximum risk of $\hat{\tau}$ is $O\left(n^{-\frac{4p}{5d-1}}\right)$, i.e. there exists $C^{(1)}$ that depends only on $\tau_{\min}$, L, f_{min}, f_{max} that
$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \le C^{(1)} n^{-\frac{4p}{5d-1}}.$$
- The minimax rate is also upper bounded by $O\left(n^{-\frac{4p}{5d-1}}\right)$, i.e.
$$\inf_{\hat{\tau}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \le C^{(1)} n^{-\frac{4p}{5d-1}}.$$

Lower bound

- The minimax rate is lower bounded by $O\left(n^{-\frac{p}{d}}\right)$, i.e. there exists $C^{(2)}$ that depends only on $\tau_{\min}$, L, f_{min}, f_{max} that
$$\inf_{\hat{\tau}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \ge C^{(2)} n^{-\frac{p}{d}}.$$

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