Estimating the reach of a manifold

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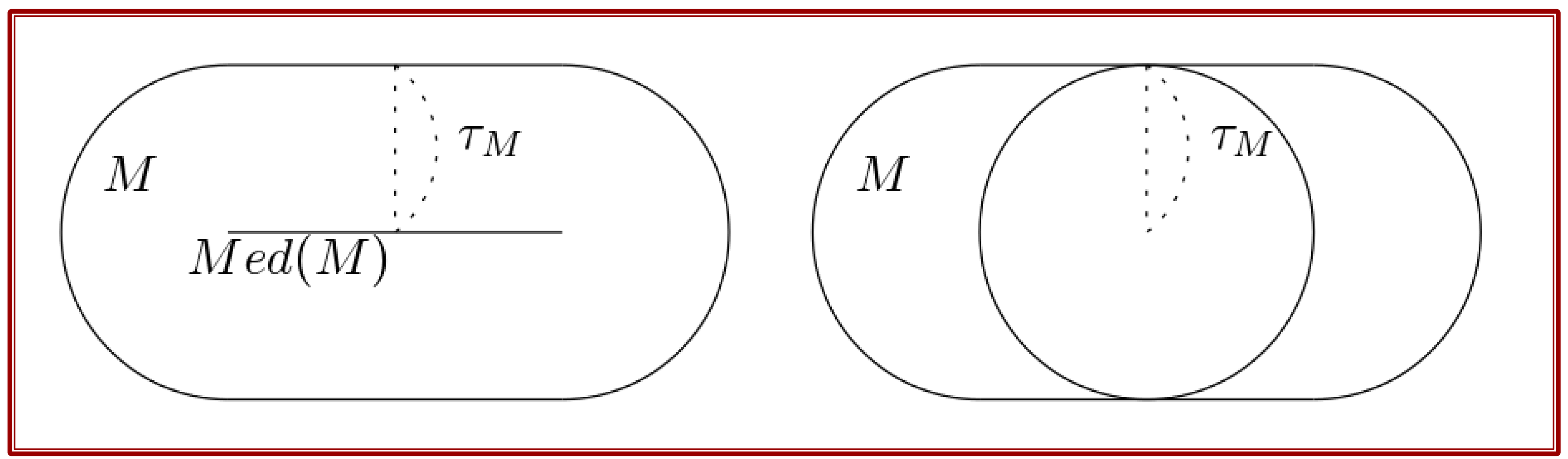
INVENTORS FOR THE DIGITAL WORLD

Abstract

Various problems within computational geometry and manifold learning encode geometric regularity through the so-called reach, a generalized convexity parameter. The reach renders a certain minimal scale of a manifold, giving bounds on both maximum curvature and possible bottleneck structures. We first study the geometry of the reach through an approximation theory perspective. Then, we proposed an estimator for the reach. Minimax upper and lower bounds are correspondingly derived.

Definition

Reach



- Medial axis of M, $Med(M)$, is the set of points in $\mathbb{R}^m$ that do not have unique nearest neighbors on M.
- The reach of M, denoted by τ_M, is the minimum distance from $Med(M)$ to M: $\tau_M = \inf_{x \in Med(M), y \in M} \|x - y\|_2$.
- The reach τ_M gives maximal offset size of M on which the projection is well defined.
- The reach τ_M also gives the maximum radius of a ball that can roll over M.

Minimax rate

- Suppose you observe iid data X and the set of distributions $\mathcal{P}$ and loss function $l(,)$ is fixed, and you are interested in parameter θ.
- Maximum risk of an estimator $\hat{\theta}$ is the risk that the estimator can make in the worst case, i.e.
$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\theta}(X), \theta(P)\right)\right].$$
- Minimax rate is the infimum of a maximum risk over all possible estimators, $\hat{\theta}$ i.e.
$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[l\left(\hat{\theta}(X), \theta(P)\right)\right].$$
- In our case, loss function is $l(a,b) = \left|\frac{1}{a} - \frac{1}{b}\right|^p$, and the parameter of interest is the reach τ_P.

Regularity conditions

We consider $\mathcal{P}$ to be set of distributions supported on d-dimensional connected compact manifolds with regular conditions:

- The reach τ_M of the support manifold M is lower bounded by $\tau_{\min}$.
- All the arc-length parametrized geodesic γ on the support manifold M has 3rd derivatives bounded by L: for all $t \in \mathbb{R}$, $\|\gamma'''(t)\| \le L$.
- The probability distribution P has density f with respect to uniform measure on M satisfying $f(x) \ge f_{\min}$.

Reach estimators

- The reach τ_M of M can be characterized as
$$\tau_M = \sup_{a,b \in M, a \neq b} \frac{\|b-a\|_2^2}{d(b-a, T_a M)}.$$

When tangent spaces are known:

- We consider the plugin estimator
$$\hat{\tau} = \sup_{1 \le i \neq j \le n} \frac{\|X_j - X_i\|_2^2}{d(X_j - X_i, T_{X_i} M)}.$$

When tangent spaces are unknown:

- $\hat{T}_i$ be an estimator of $T_{X_i} M$, then we consider the plugin estimator
$$\hat{\tau} = \sup_{1 \le i \neq j \le n} \frac{\|X_j - X_i\|_2^2}{d(X_j - X_i, \hat{T}_i)}.$$

Upper bound and Lower bound of Minimax Rate

Upper bound

- The maximum risk of $\hat{\tau}$ is $O\left(n^{-\frac{2p}{3d-1}}\right)$, i.e. there exists $C^{(1)}$ that depends only on $\tau_{\min}$, L, f_{min}, f_{max} that
$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \le C^{(1)} n^{-\frac{2p}{3d-1}}.$$
- The minimax rate is also upper bounded by $O\left(n^{-\frac{2p}{3d-1}}\right)$, i.e.
$$\inf_{\hat{\tau}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \le C^{(1)} n^{-\frac{2p}{3d-1}}.$$

Lower bound

- The minimax rate is lower bounded by $O\left(n^{-\frac{p}{d}}\right)$, i.e. there exists $C^{(2)}$ that depends only on $\tau_{\min}$, L, f_{min}, f_{max} that
$$\inf_{\hat{\tau}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}\left[\left|\frac{1}{\hat{\tau}} - \frac{1}{\tau_M}\right|^p\right] \ge C^{(2)} n^{-\frac{p}{d}}.$$

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